An interesting family of Black-Scholes perpetuities

Bernard Roynette, Marc Yor

An interesting family of Black-Scholes perpetuities

B. Roynette[(1)], M. Yor[(2), (3)]

12/11/2008

[(1)] Institut Elie Cartan, Université Henri Poincaré,
B.P. 239, 54506 Vandoeuvre les Nancy Cedex
email : bernard.roynette@iecn.u-nancy.fr

[(2)] Laboratoire de Probabilités et Modèles Aléatoires,
Université Paris VI et VII, 4 place Jussieu - Case 188
F - 75252 Paris Cedex 05
email : deaproba@proba.jussieu.fr

[(3)] Institut Universitaire de France

Abstract We obtain the Laplace transform and integrability properties of the integral over $\mathbb{R}_+$ of the call quantity associated with geometric Brownian motion with negative drift, thus adding a new element to the list of already studied Brownian perpetuities.

Key words Bessel-McDonald functions, Sturm-Liouville equation, perpetuities.

AMS Classification 60 J 25, 60 J 65, 60 G 44.

1 Motivation, introduction

1.1 Let $(B_t,\ t \geq 0)$ denote 1-dimensional Brownian motion starting from 0 ; associated to B, one considers the geometric Brownian motion :

$$\left(\mathcal{E}_t := \exp\left(B_t - \frac{t}{2}\right),\ t \geq 0\right)$$

which is a positive martingale converging a.s. to 0 as $t \to \infty$. In recent years, the following questions have been asked to the second author, in connection with European option pricing :

i) to express as simply as possible the quantity :

$$\int_0^\infty \lambda\, e^{-\lambda t}\, E\big((\mathcal{E}_t - k)^+\big) dt \qquad (k \geq 0)$$

ii) to find the law of $\displaystyle\int_0^s (\mathcal{E}_t - k)^+ dt$, for fixed s

iii) to find the law of $\displaystyle\int_0^\infty \lambda\, e^{-\lambda t} (\mathcal{E}_t - k)^+ dt$

i) may be solved quite explicitly, and indeed this has led the authors, jointly with D. Madan, to write a series of papers relating European option prices and laws of last passage times of continuous positive martingales converging to 0 as $t \to \infty$ $\big($see, e.g. [MRY1], [MRY2], [MRY3]$\big)$.
Questions *ii)* and *iii)* are harder to solve explicitly, as, indeed, one may look for double Laplace transforms of either quantities, e.g. :

$$\int_0^\infty e^{-\mu t} E(e^{-\lambda \int_0^t (\mathcal{E}_s - k)^+ ds}) dt$$

In the present paper, we are studying thoroughly the law of $\displaystyle\int_0^\infty (\mathcal{E}_t - k)^+ dt$ which seems to be slightly less difficult than either question *ii)* or *iii)*. However, the results we obtain are not particulary simple.

1.2 These motivations having been presented, we now concentrate exclusively on the law of $\displaystyle\int_0^\infty (\mathcal{E}_t - k)^+ dt$. It will be convenient to consider the two-parameter process :

$$\left(\mathcal{E}_t^{(x)} := x \exp\left(B_t - \frac{t}{2}\right) = \exp\left(\log x + B_t - \frac{t}{2}\right),\ t \geq 0\right) \tag{1.1}$$

$(\mathcal{E}_t^{(x)},\ t \geq 0,\ x > 0)$ is a Markov process taking values in $\mathbb{R}_+$ which, most often, we shall denote as : $\big((\mathcal{E}_t,\ t \geq 0\ ;\ P_x,\ x > 0\big)$ since $\mathcal{E}_0^{(x)} = x$.
To simplify notation, we shall write P for P_1 and $\mathcal{E}_t$ for $\mathcal{E}_t^{(1)}$.
For any $k > 0$, let us define :

$$\Pi_k^{(x)} := \int_0^\infty (\mathcal{E}_t^{(x)} - k)^+ dt \tag{1.2}$$

Again, to simplify, we shall write Π_k for $\Pi_k^{(1)}$. Since $\mathcal{E}_t^{(x)} \underset{t\to\infty}{\longrightarrow} 0$ a.s., the integral which defines (1.2) is a.s. convergent, as $(\mathcal{E}_t^{(x)} - k)^+ = 0$ for $t \geq \mathcal{G}_k := \sup\{t\ ;\ \mathcal{E}_t^{(x)} = k\}$, and $\mathcal{G}_k < \infty$ a.s.

1.3 We now explain about some reductions of the study of the laws of $\Pi_k^{(x)}$, to closely related problems.
i) From Itô's formula, we deduce :

$$\mathcal{E}_t^{(x)} = x + \int_0^t \mathcal{E}_s^{(x)} dB_s = \beta_{A_t^{(x)}} \tag{1.3}$$

where $(\beta_u,\ u \geq 0)$ denotes the Dubins-Schwarz Brownian motion associated with $(\mathcal{E}_t^{(x)},\ t \geq\ 0)$ (and $\beta_0 = x$) and :

$$A_t^{(x)} = <\mathcal{E}^{(x)}>_t = \int_0^t (\mathcal{E}_s^{(x)})^2 ds \tag{1.4}$$

Hence :

$$\Pi_k^{(x)} = \int_0^\infty (\beta_{A_s^{(x)}} - k)^+ ds = \int_0^{T_0(\beta)} \frac{(\beta_v - k)^+}{\beta_v^2}\, dv \tag{1.5}$$

$\big($after the change of variable $A_s^{(x)} = v$ and where $T_0(\beta) = \inf\{u \geq 0\ ;\ \beta_u = 0\}\big)$

$$= \int_k^\infty \frac{(y-k)}{y^2}\, L^y_{T_0(\beta)}\, dy \qquad \left(= \int_k^\infty dz \int_z^\infty \frac{dy}{y^2}\, L^y_{T_0(\beta)} \right) \tag{1.6}$$

from the density of occupation formula, and where $L^y_{T_0}$ denotes the local time at time T_0 and level y of Brownian motion $(\beta_u,\ u \geq 0)$.

ii) When $x = k$, the first Ray-Knight Theorem allows us to write, from (1.6) :

$$\Pi_k^{(k)} = \int_k^\infty \frac{(y-k)}{y^2}\, \lambda_{y-k}\, dy \tag{1.7}$$

where $(\lambda_z,\ z \geq 0)$, conditionally on $\lambda_0 = l$, is a 0-dimensional squared Bessel process starting at l, and where λ_0 is an exponential variable with parameter $\frac{1}{2k}$, i.e. with expectation $2k$.

1.4 From the elementary relations :

$$(x\, e^{B_t - \frac{t}{2}} - k)^+ = x \left(e^{B_t - \frac{t}{2}} - \frac{k}{x} \right)^+ = k \left(\frac{x}{k} \big(e^{B_t - \frac{t}{2}} \big) - 1 \right)^+ \tag{1.8}$$

valid for every $x, k > 0$, we deduce that the law of Π_k under P_x is that of $x\, \Pi_{\frac{k}{x}}$ under P, or that of $k\Pi_1$ under $P_{\frac{x}{k}}$. In other words, for every Borel positive function φ, we have :

$$E_x\big[\varphi(\Pi_k)\big] = E\big[\varphi(x\, \Pi_{\frac{k}{x}})\big] = E_{\frac{x}{k}}\big(\varphi(k\Pi_1)\big) \tag{1.9}$$

These relations allow us to reduce our study of the law of Π_k under P_x to that of Π_1 under $P_{\frac{x}{k}}$. We might as well limit ourselves to the study of Π_k under P.

1.5 It is proven in Dufresne [Duf] and Yor [Y] $\big($see also P. Salminen and M. Yor [SY1]$\big)$ that, for every $a \neq 0$ and $\nu > 0$:

$$\int_0^\infty \exp\,(aB_t - \nu t) dt \overset{\text{(law)}}{=} \frac{2}{a^2 \gamma_{\frac{2\nu}{a^2}}} \tag{1.10}$$

where γ_b denotes a gamma variable with parameter b, i.e. :

$$P(\gamma_b \in dt) = \frac{1}{\Gamma(b)}\, e^{-t}\, t^{b-1} dt \qquad (t \geq 0) \tag{1.11}$$

In particular, for $a = 1$ and $\nu = \dfrac{1}{2}$

$$\Pi_0 \overset{\text{(law)}}{=} \frac{2}{\gamma_1} = \frac{2}{\mathfrak{e}} \tag{1.12}$$

where $\mathfrak{e}$ is a standard exponential variable.

1.6 Let $\nu \neq 0$ and :

$$\mathcal{E}_t^{(x,\nu)} := \exp\left(\nu\left(\log x + B_t\right) - \frac{\nu^2 t}{2}\right) = x^\nu \exp\left(\nu B_t - \frac{\nu^2 t}{2}\right)$$

and define, for $k > 0$:

$$\Pi_k^{(x,\nu)} := \int_0^\infty (\mathcal{E}_s^{(x,\nu)} - k)^+ ds \tag{1.13}$$

Since, by scaling, $(\mathcal{E}_t^{(x,\nu)},\ t \geq 0) \overset{\text{(law)}}{=} (x^\nu \mathcal{E}_{\nu^2 t},\ t \geq 0)$, we have :

$$\begin{aligned}\Pi_k^{(x,\nu)} &\overset{\text{(law)}}{=} \int_0^\infty (x^\nu \mathcal{E}_{\nu^2 s} - k)^+ ds = \frac{x^\nu}{\nu^2}\int_0^\infty \left(\mathcal{E}_v - \frac{k}{x^\nu}\right)^+ dv \\ &\overset{\text{(law)}}{=} \frac{x^\nu}{\nu^2}\,\Pi_{\frac{k}{x^\nu}}\end{aligned} \tag{1.14}$$

Thus, the study of the law of $\Pi_k^{(x,\nu)}$ may be reduced very simply to that of $\Pi_{\frac{k}{x^\nu}}$. This is the reason why we have chosen, in this paper, to limit ourselves to $\nu = 1$.

1.7 Here are our results :

Theorem 1.1 *Let $\alpha \geq 0$. Then, for every $x > 0$, $E_x\big((\Pi_1)^\alpha\big) < \infty$ if and only if $\alpha < 1$*

Theorem 1.2 *Let $\alpha < 0$ and $x > 0$. Then :*

i) For every $x > 1$, $E_x\big((\Pi_1)^\alpha\big) < \infty$.

ii) For every $x < 1$, $P_x(\Pi_1 = 0) = 1 - x$; hence $E_x\big((\Pi_1)^\alpha\big) = +\infty$.

iii) For $x = 1$, $E_1\big((\Pi_1)^\alpha\big) < \infty$ if and only if $|\alpha| < \dfrac{1}{3}$.

Theorem 1.3 *(Laplace transform of Π_1). For every $\theta \geq 0$:*

$$E_x(e^{-\frac{\theta}{2}\Pi_1}) = \begin{cases} \dfrac{\sqrt{x}\,K_\gamma(\sqrt{4\theta x})}{\frac{1}{2}K_\gamma(\sqrt{4\theta}) - \sqrt{\theta}\,K'_\gamma(\sqrt{4\theta})} & \text{if } x \geq 1 \\ 1 + x\,\dfrac{\frac{1}{2}K_\gamma(\sqrt{4\theta}) + \sqrt{\theta}K'_\gamma(\sqrt{4\theta})}{\frac{1}{2}K_\gamma(\sqrt{4\theta}) - \sqrt{\theta}K'_\gamma(\sqrt{4\theta})} & \text{if } 0 < x \leq 1 \end{cases} \tag{1.15}$$

where K_γ denotes the Bessel-McDonald function with index γ $\big($see [Leb], p. 108$\big)$ and where $\gamma = \sqrt{1-4\theta}$ if $4\theta \leq 1$ and $\gamma = i\sqrt{4\theta - 1}$ if $4\theta \geq 1$.
We shall also prove, as a consequence of Theorem 1.3 :

Theorem 1.4 *Let $(\lambda_x,\ x \geq 0)$ denote a squared Bessel process with dimension 0 started at l and denote its law by $Q_l^{(0)}$. Then :*
- *If $4\theta \geq 1$:*

$$\begin{aligned} Q_l^{(0)} \left(\exp -\frac{\theta}{2} \int_1^\infty \frac{(x-1)}{x^2}\ \lambda_{x-1} dx \right) & \left(= Q_l^{(0)} \left(\exp -\frac{\theta}{2} \int_0^\infty \frac{x}{(x+1)^2}\ \lambda_x dx \right) \right. \\ &= \exp\ \frac{l}{2} \left(\frac{1}{2} + \frac{\sqrt{\theta} K'_{i\nu}(\sqrt{4\theta})}{K_{i\nu}(\sqrt{4\theta})} \right) \end{aligned} \tag{1.16}$$

with $\nu = \sqrt{4\theta - 1}$
- *If $4\theta \leq 1$:*

$$Q_l^{(0)} \left(\exp -\frac{\theta}{2} \int_1^\infty \frac{x-1}{x^2}\ \lambda_{x-1}\, dx \right) = \exp \frac{l}{2} \left(\frac{1}{2} + \frac{\sqrt{\theta}\, K'_\nu(\sqrt{4\theta})}{K_\nu(\sqrt{4\theta})} \right)$$

with $\nu = \sqrt{1-4\theta}$.

In Section 3 of this work, we shall study the asymptotic behavior, as $\theta \to \infty$, of $E(e^{-\frac{\theta}{2}\Pi_1})$ and we shall obtain :

$$E(e^{-\frac{\theta}{2}\Pi_1}) \underset{\theta\to\infty}{\sim} \frac{C}{\theta^{\frac{1}{3}}} \qquad (C > 0) \tag{1.17}$$

Finally, in a short Section 4, we shall indicate how Theorems 1.1, 1.2 and 1.3 extend when we replace Π_k by $\Pi_k^{(\rho)}$, with :

$$\begin{aligned} \Pi_k^{(\rho)} \quad &:= \quad \int_0^\infty \left(e^{\rho(B_t - \frac{t}{2})} - k \right)^+ dt \qquad (\rho, k > 0) \\ &\overset{(\text{law})}{=} \quad \frac{1}{\rho^2} \int_0^\infty (e^{(B_u - \frac{u}{2\rho})} - k)^+ du \qquad (\text{by scaling}) \end{aligned} \tag{1.18}$$

which, from (1.18), corresponds to consider an extension of our previous perpetuities relative to $\left(B_t - \dfrac{t}{2},\ t \geq 0\right)$ to Brownian motion with drift $-\dfrac{1}{2\rho}$, i.e. $\left(B_t - \dfrac{t}{2\rho},\ t \geq 0\right)$.

1.8 For studies of other perpetuities related to Brownian motion with drift, we refer the reader to Salminen-Yor $\big($[SY1], [SY2], [SY3]$\big)$.

2 Proofs of Theorems 1.1, 1.2 and 1.3

2.1 A first proof of Theorem 1.1

2.1 *i)* We now prove that, for every $x>0$, $E_x(\Pi_1)=\infty$
Indeed, from (1.6) :

$$\begin{aligned} E_1(\Pi_1) &= \int_1^\infty \frac{(y-1)}{y^2}\, E_1(L^y_{T_0})dy \\ &= \int_1^\infty \frac{(y-1)}{y^2}\, E_1(L^1_{T_0})dy \\ &= 2\int_1^\infty \frac{(y-1)}{y^2}\, dy = +\infty \end{aligned} \tag{2.1}$$

$\Big($since, for $y \geq 1$, $E_1\big(L^y_{T_0}\big) = E_1\big(L^1_{T_0}\big) = 2$, as $L^1_{T_0}$ is an exponential variable with parameter $\frac{1}{2}\Big)$.

2.1 *ii)* We now prove that, for every $x>0$, $E_x(\Pi_1)=\infty$

- For $x \geq 1$, this is clear since $\Pi^{(x)}_k$ is an increasing function of x (and a decreasing function of k). Hence :

$$E_x(\Pi_1) \geq E_1(\Pi_1) = +\infty \qquad \big(\text{from } (2.1)\big)$$

- For $x<1$, from the Markov property :

$$\Pi^{(x)}_1 \overset{\text{(law)}}{=} 1_{\left\{T^{(-\frac{1}{2})}_{\log(\frac{1}{x})}<\infty\right\}} \cdot \Pi_1 \tag{2.2}$$

with $T^{(-\frac{1}{2})}_{\log(\frac{1}{x})} := \inf\left\{t\geq 0\ ;\ B_t - \dfrac{t}{2} = \log\left(\dfrac{1}{x}\right)\right\}$, and in (2.2), Π_1 and $T^{(-\frac{1}{2})}_{\log(\frac{1}{x})}$ are assumed independent. Hence :

$$E(\Pi^{(x)}_1) = P(T^{(-\frac{1}{2})}_{\log(\frac{1}{x})}<\infty)\ E(\Pi_1) = +\infty$$

from (2.1) and since $P(T^{(-\frac{1}{2})}_{\log(\frac{1}{x})}<\infty) = x$ (see Lemma 2.1 below).

2.1 *iii)* We prove that, for $0\leq\alpha<1$, $E_x\big((\Pi_1)^\alpha\big)<\infty$
We have :

$$\begin{aligned} E_x\big((\Pi_1)^\alpha\big) &= x^\alpha\, E\big((\Pi_{\frac{1}{x}})^\alpha\big) \quad \big(\text{from } (1.9)\big) \\ &\leq x^\alpha\, E\big((\Pi_0)^\alpha\big) \end{aligned}$$

(since Π_k is a decreasing function of k)

$$\leq x^\alpha \int_0^\infty \frac{2}{y^\alpha}\, e^{-y}dy < \infty \qquad \big(\text{from} (1.12)\big)$$

2.2 Second proof of Theorem 1.1

It hinges upon :

Lemma 2.1

Let $G_a^{(\nu)} := \sup\{u \geq 0\ ;\ B_u + \nu u = a\}$ *(*$= 0$ *if this set is empty) and*

$T_a^{(\nu)} := \inf\{u \geq 0\ ;\ B_u + \nu u = a\}$ *(*$= +\infty$ *if this set is empty).*

i) If ν *and* a *have the same sign :*

- $E\big(e^{\mu G_a^{(\nu)}}\big) < \infty$ *if and only if* $\mu < \dfrac{\nu^2}{2}$ (2.3)
- *for every real* α, $E\big((G_a^{(\nu)})^\alpha\big) < \infty$ (2.4)

ii) If ν *and* a *have opposite signs :*

- $E\big(e^{\mu G_a^{(\nu)}}\big) < \infty$ *if and only if* $\mu < \dfrac{\nu^2}{2}$ (2.5)
- *for every real* $\alpha > 0$, $E\big((G_a^{(\nu)})^\alpha\big) < \infty$ (2.6)
- $P(T_a^{(\nu)} = \infty) = P(G_a^{(\nu)} = 0) = 1 - e^{2\nu a}$ (2.7)

Hence, for $\alpha < 0$, $E\big((G_a^{(\nu)})^\alpha\big) = +\infty$

The proof of this Lemma is obvious. It hinges on the well-known formulae :

• If $\nu, a > 0$:

$$P(G_a^{(\nu)} \in dt) = \frac{\nu}{\sqrt{2\pi t}}\ \exp\left(-\frac{1}{2t}(a - \nu t)^2\right) dt$$

• If $\nu, a > 0$:

$$P(G_a^{(-\nu)} \in dt) = \frac{\nu}{\sqrt{2\pi t}}\ \exp\left(-\frac{1}{2t}(a + \nu t)^2\right) dt$$

• If ν and a have opposites signs :

$$P(G_a^{(\nu)} > 0) = P(T_a^{(\nu)} < \infty) = e^{2\nu a}$$

• $G_a^{(\nu)} \overset{\text{(law)}}{=} G_{(-a)}^{(-\nu)} \overset{\text{(law)}}{=} \dfrac{1}{T_\nu^{(a)}}$ ■

We now give a second proof of Theorem 1.1.

• We first show that, for $0 \leq \alpha < 1$, $E_x\big((\Pi_1)^\alpha\big) < \infty$. From the relation :

$$\begin{aligned}\Pi_1^{(x)} &= x\int_0^\infty \left(\mathcal{E}_t - \frac{1}{x}\right)^+ dt = x\int_0^{G_{-\log x}^{\left(-\frac{1}{2}\right)}} \left(\mathcal{E}_t - \frac{1}{x}\right)^+ dt \\ &\leq x\left(\sup_{t\geq 0}\mathcal{E}_t\right)\cdot G_{-\log x}^{\left(-\frac{1}{2}\right)}\end{aligned}$$

we deduce that, for $0 \le \alpha < 1$, $\alpha p < 1$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$ $(p, q > 1)$:

$$\begin{aligned} E_x\big((\Pi_1)^\alpha\big) &\le x^\alpha\, E\left[\left(\sup_{t\ge 0} \mathcal{E}_t\right)^\alpha \left(G^{(-\frac{1}{2})}_{-\log x}\right)^\alpha\right] \\ &\le x^\alpha\, E\left(\left(\sup_{t\ge 0} \mathcal{E}_t\right)^{\alpha p}\right)^{\frac{1}{p}} E\left(\left(G^{(-\frac{1}{2})}_{(-\log x)}\right)^{\alpha q}\right)^{\frac{1}{q}} \\ &\le C\, x^\alpha\, E\left(\left(\sup_{t\ge 0} \mathcal{E}_t\right)^{\alpha p}\right)^{\frac{1}{p}} \qquad \text{(from Lemma 2.1)} \end{aligned}$$

As $(\mathcal{E}_t,\ t \ge 0)$ is a positive martingale, starting from 1, and converging a.s. to 0 as $t \to \infty$, we get $\big($see [NY] or [RY], Chap. II, Ex. 3.12, p. 73$\big)$:

$$\sup_{s\ge 0} \mathcal{E}_s \overset{\text{(law)}}{=} \frac{1}{U} \tag{2.8}$$

with U uniform on $[0,1]$. Thus :

$$E\left(\left(\sup_{t\ge 0} \mathcal{E}_t\right)^{\alpha p}\right) = \int_0^1 \frac{1}{u^{\alpha p}}\, du < \infty \tag{2.9}$$

since $\alpha p < 1$.

- We then show that $E_x(\Pi_1) = \infty$. First of all, it follows from [MRY1] that :

$$E_1\big((\mathcal{E}_t - k)^+\big) = P\big(G^{(\frac{1}{2})}_{\log k} \le t\big) \tag{2.10}$$

Hence :

$$E_1(\Pi_1) = \int_0^\infty P\big(G^{(\frac{1}{2})}_0 \le t\big) dt = E\left(\int_{G^{(\frac{1}{2})}_0}^\infty dt\right) = +\infty$$

since $G^{(\frac{1}{2})}_0 < +\infty$ a.s.
Likewise :

$$\begin{aligned} E_x(\Pi_1) &= x\, E_1(\Pi_{\frac{1}{x}}) \qquad \big(\text{from (1.9)}\big) \\ &= x\, E\left(\int_{G^{(\frac{1}{2})}_{\log \frac{1}{x}}}^\infty dt\right) = \infty \end{aligned}$$

from (2.10) and since $G^{(\frac{1}{2})}_{\log \frac{1}{x}} < \infty$ a.s.

2.3 Proof of Theorem 1.2

i) We already prove point *i)* :

Let $x > 1$ and x' such that $1 < x' < x$. It is then obvious that :

$$\Pi_1^{(x)} \geq (x'-1) T^{(-\frac{1}{2})}_{\log(\frac{x'}{x})} \tag{2.11}$$

$\big($with $T_a^{(-\frac{1}{2})} := \inf\{t \geq 0 \,;\; B_t - \frac{t}{2} = a\}$, since, if $t < T^{(-\frac{1}{2})}_{\log\frac{x'}{x}}$, then $e^{\log x + B_t - \frac{t}{2}} - 1 \geq x' - 1\big)$.

Hence, with $\gamma > 0$:

$$\begin{aligned}
E_x\left(\frac{1}{(\Pi_1)^\gamma}\right) &= \int_0^\infty P_x\left(\frac{1}{(\Pi_1)^\gamma} \geq t\right) dt = \int_0^\infty P_x(\Pi_1 < v) \frac{\gamma}{v^{1+\gamma}}\, dv \\
&\leq \int_0^\infty \frac{\gamma}{v^{1+\gamma}}\, dv\, P\left(T^{(-\frac{1}{2})}_{\log\frac{x'}{x}} \leq \frac{v}{x'-1}\right) dv \qquad \big(\text{from (2.11)}\big) \\
&= \int_0^\infty \frac{\gamma}{v^{1+\gamma}}\, dv \left(P\big(G_{\frac{1}{2}}^{(\log(\frac{x}{x'}))} \geq \frac{x'-1}{v}\right) dv \\
\Big(\text{since} \qquad & T_a^{(\nu)} \overset{\text{(law)}}{=} \frac{1}{G_\nu^{(a)}} \quad \text{and} \quad G_\nu^{(a)} \overset{\text{(law)}}{=} G_{-\nu}^{(-a)}\Big) \\
&= \int_0^\infty \frac{\gamma}{(x'-1)^\gamma}\, u^{\gamma-1}\, du\, P\big(G_{\frac{1}{2}}^{(\log(\frac{x}{x'}))} \geq u\big) du \\
&= \frac{1}{(x'-1)^\gamma}\, E\big((G_{\frac{1}{2}}^{(\log\frac{x}{x'})})^\gamma\big) < \infty
\end{aligned} \tag{2.12}$$

$\big($from point *i)* of Lemma 2.1$\big)$.

ii) We now prove point *ii)* :

It is clear that, for $x < 1$:

$$\{\Pi_1^{(x)} = 0\} = \big\{T^{(-\frac{1}{2})}_{\log\frac{1}{x}} = \infty\big\}$$

Thus, $P\{\Pi_1^{(x)} = 0\} = P\big\{T^{(-\frac{1}{2})}_{\log\frac{1}{x}} = \infty\big\} = P\big\{G_{\frac{1}{2}}^{(\log x)} = 0\big\} = 1 - x > 0$ from Lemma 2.1 and (1.26).

iii) We now prove point *iii)* :

For this purpose, we write, for $\gamma > 0$:

$$E_1\left(\frac{1}{(\Pi_1)^\gamma}\right) = \frac{1}{\Gamma(\gamma)} \int_0^\infty E(e^{-\theta\,\Pi_1}) \theta^{\gamma-1} d\theta \tag{2.13}$$

and we show, in the next Section 3, that :

$$E_1(e^{-\theta\Pi_1}) \underset{\theta\to\infty}{\sim} \frac{C}{\theta^{\frac{1}{3}}} \qquad (C > 0)$$

Thus, $E\left(\frac{1}{(\Pi_1)^\gamma}\right) < \infty$ if and only if $\int_1^\infty \theta^{\gamma-1-\frac{1}{3}} d\theta < \infty$, that is, if and only if $\gamma < \frac{1}{3}$.

2.4 Proof of Theorem 1.3

a) **A useful Lemma**

Lemma 2.2 *Let $\theta \geq 0$ and K_γ the Bessel McDonald function with index γ, such that $\gamma = \sqrt{1-4\theta}$ if $\theta \leq \frac{1}{4}$ and $\gamma = i\sqrt{4\theta - 1}$ if $\theta \geq \frac{1}{4}$.*

1) *Define the function $\varphi_\theta : \mathbb{R}_+ \to \mathbb{R}$ by :*

$$\varphi_\theta(y) = \sqrt{y}\, K_\gamma(\sqrt{4\theta y}) \qquad (y \geq 0) \tag{2.14}$$

Then :

i) φ_θ is a real valued function which satisfies :

$$\varphi_\theta''(y) + \left(-\frac{\theta}{y} + \frac{\theta}{y^2}\right)\varphi_\theta(y) = 0 \tag{2.15}$$

ii) φ_θ, restricted to $[1, \infty[$ is positive, convex, bounded and decreasing.

2) *We define the function $\widetilde{\varphi}_\theta : \mathbb{R}_+ \to \mathbb{R}_+$ by :*

$$\widetilde{\varphi}_\theta(y) = \begin{cases} \varphi_\theta(y) & if\ y \geq 1 \\ \big(\varphi_\theta(1) - \varphi_\theta'(1)\big) + y\varphi_\theta'(1) & if \quad 0 \leq y \leq 1 \end{cases} \tag{2.16}$$

Then $\widetilde{\varphi}_\theta$ is a bounded, positive, convex, decreasing function which satisfies :

$$\widetilde{\varphi}_\theta''(y) + \left(-\frac{\theta}{y} + \frac{\theta}{y^2}\right) 1_{y \geq 1} \cdot \widetilde{\varphi}_\theta(y) = 0 \tag{2.17}$$

b) **Proof of Lemma 2.2**

i) Relation (2.15) $\big($as well as relation (2.17)$\big)$ follows from a direct computation, using the equation $K_\gamma''(z) + \frac{1}{z}\, K_\gamma'(z) - \left(1 + \frac{\gamma^2}{z^2}\right) K_\gamma(z) = 0$ $\big($see [Leb], p. 110$\big)$ and the fact that $\gamma^2 = 1 - 4\theta$ $\big($see Petiau [Pet], p. 306, formula (8), which needs to be corrected by replacing a by $-a$, or Kamke [Kam], p. 440$\big)$.

We distinguish two cases :

Case 1 : $4\theta \leq 1$, $\gamma = \sqrt{1-4\theta}$. The function K_γ, hence also φ_θ, is positive. Furthermore φ_θ is bounded on $\mathbb{R}_+$ since :

$$\varphi_\theta(y) \underset{y \to 0}{\sim} C\, y^{\frac{1-\gamma}{2}}, \ \varphi_\theta(y) \underset{y \to \infty}{\sim} C'\, y^{\frac{1}{4}}\, e^{-\sqrt{4\theta y}} \qquad \big(\text{see [Leb], p. 123 and 136}\big)$$

On the other hand, from (2.15), the function φ_θ is convex on the interval $[1, \infty[$. As it is convex, positive, and bounded, it is decreasing. Lemma 2.2 is thus proven in this case.

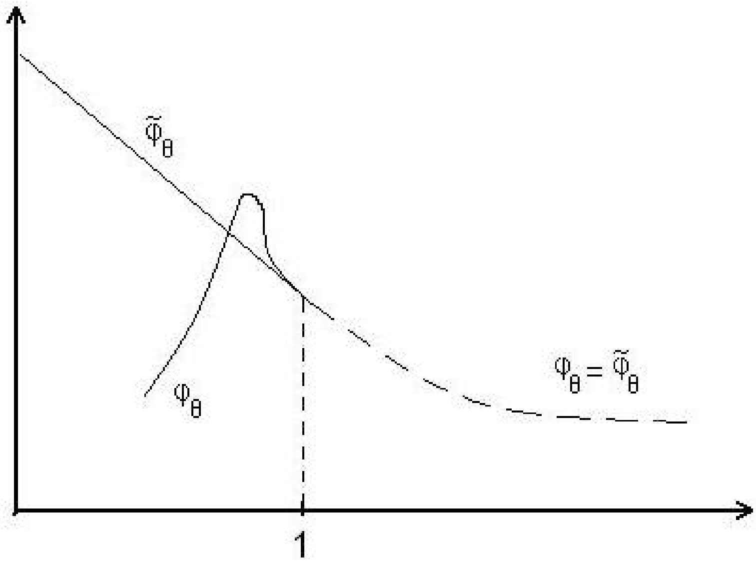

Figure 1: Graphs of φ_θ and $\widetilde{\varphi}_\theta$

Case 2 : $4\theta \geq 1$, $\gamma = i\sqrt{4\theta - 1}$ and here $\varphi_\theta(y) = \sqrt{y}\, K_{i\nu}(\sqrt{4\theta y})$ with $\nu = \sqrt{4\theta - 1}$. From the integral representation formula (see [Leb], p. 119) :

$$K_{i\nu}(y) = \int_0^\infty e^{-y\cosh u} \cos(\nu u) du \qquad (y \geq 0) \tag{2.18}$$

we deduce that $K_{i\nu}(y)$ is real valued, hence so is $\varphi_\theta(y)$.
On the other hand, φ_θ is bounded on $[0, \infty[$. Indeed, for $y \geq 0$, from (2.18) :

$$\left|\sqrt{4\theta}\, \varphi_\theta\left(\frac{y^2}{4\theta}\right)\right| = y\big|K_{i\nu}(y)\big| \leq y \int_0^\infty e^{-y\frac{e^u}{2}} du \leq y \int_{\frac{y}{2}}^\infty e^{-v} \frac{dv}{v}$$

Thus :

$$\left|\sqrt{4\theta}\, \varphi_\theta\left(\frac{y^2}{4\theta}\right)\right| \leq e^{-\frac{y}{4}} \qquad \text{for } y \text{ large enough, hence}$$

$$\left|\sqrt{4\theta}\, \varphi_\theta\left(\frac{y^2}{4\theta}\right)\right| \underset{y\to\infty}{\longrightarrow} 0 \qquad \text{and}$$

$$\left|\sqrt{4\theta}\, \varphi_\theta\left(\frac{y^2}{4\theta}\right)\right| \leq y\left(C + |\log(y)|\right) \underset{y\to 0}{\longrightarrow} 0$$

On the other hand, it is clear, from (2.18), that $K_{i\nu}(y) > 0$ hence $\varphi_\theta(y) > 0$, for y large enough.

• We now show that $K_{i\nu}$ is decreasing on $[\nu, \infty[$. If not, there would exist a point $y_0 > \nu$ such that :

$$K_{i\nu}(y_0) > 0,\ K'_{i\nu}(y_0) = 0,\ K''_{i\nu}(y_0) \leq 0.$$

However,

$$K''_{i\nu}(y_0) + \frac{1}{y_0}\, K'_{i\nu}(y_0) = \left(1 - \frac{\nu^2}{y_0^2}\right) K_{i\nu}(y_0)$$

Thus $K''_{i\nu}(y_0) > 0$, which is absurd. Since $K_{i\nu}$ is decreasing on $[\nu, \infty[$, and positive near ∞, $K_{i\nu}$ is positive on $[\nu, \infty[$. Thus, φ_θ is positive on $[1, \infty[$ $\big($since, if $y \geq 1$, $\sqrt{4\theta y} \geq \sqrt{4\theta} \geq \sqrt{4\theta - 1} = \nu\big)$. From the relation (2.15), we then deduce that φ_θ is convex on $[1, \infty[$. Since it is bounded, convex and positive, it is decreasing. Lemma 2.2 is proven. ■

c) **End of the proof of Theorem 1.3.**

Let $\left(M_t^\theta := \widetilde{\varphi}_\theta(B_t) \exp -\frac{\theta}{2} \int_0^t \frac{\widetilde{\varphi}''_\theta}{\varphi_\theta}(B_s)ds,\ t \geq 0\right)$. Then $(M_t^\theta,\ t \geq 0)$ is a local martingale. It is equal, from (2.17) to :

$$M_t^\theta = \widetilde{\varphi}_\theta(B_t) \exp -\frac{\theta}{2} \int_0^t \frac{(B_s - 1)^+}{B_s^2}\, ds \tag{2.19}$$

and, from Lemma 2.2, for every $x \geq 0$, $(M_{t \wedge T_0}^\theta,\ t \geq 0)$ is bounded. Thus, from Doob's optional stopping Theorem :

$$\begin{aligned} \widetilde{\varphi}_\theta(x) &= E_x(M_0^\theta) = E_x\left[\widetilde{\varphi}_\theta(B_{T_0}) \exp -\frac{\theta}{2} \int_0^{T_0(B)} \frac{(B_s - 1)^+}{B_s^2} ds\right] \\ &= \widetilde{\varphi}_\theta(0) E_x\left[\exp -\frac{\theta}{2}\, \Pi_1\right] \qquad \big(\text{from } (1.5)\big) \end{aligned}$$

Thus :

$$E_x\big(e^{-\frac{\theta}{2}\Pi_1}\big) = \frac{\widetilde{\varphi}_\theta(x)}{\widetilde{\varphi}_\theta(0)}$$

This is precisely Theorem 1.3, owing to formula (2.16) which yields $\widetilde{\varphi}_\theta$ explicitly. ■

Remark 2.3 Theorem 1.3 allows to recover formula (1.9) : $\Pi_0 \overset{\text{(law)}}{=} \frac{2}{\mathfrak{e}}\cdot$ Indeed, on one hand :

$$E_1\big(e^{-\frac{\theta}{2}\Pi_0}\big) = E\big(e^{-\frac{\theta}{2}\frac{2}{\mathfrak{e}}}\big) = \int_0^\infty e^{-\frac{\theta}{z} - z} dz = 2\sqrt{\theta}\, K_1(\sqrt{4\theta}) \qquad \big(\text{see [Leb], p. 119}\big)$$

and, on the other hand :

$$
\begin{aligned}
E_1(e^{-\frac{\theta}{2}\Pi_0}) &= \lim_{\epsilon\downarrow 0} E_1(e^{-\frac{\theta}{2}\Pi_\epsilon}) = \lim_{\epsilon\downarrow 0} E\left(\exp -\frac{\theta}{2}\int_0^\infty (e^{B_t-\frac{t}{2}}-\epsilon)^+ dt\right) \\
&= \lim_{\epsilon\downarrow 0} E\big(e^{-\frac{\theta\epsilon}{2}\int_0^\infty (\frac{1}{\epsilon}(e^{B_t-\frac{t}{2}}-1)^+ dt}\big) = \lim_{\epsilon\downarrow 0} E_{\frac{1}{\epsilon}}(e^{-\frac{\theta\epsilon}{2}\Pi_1}) \\
&= \lim_{\epsilon\downarrow 0} \frac{\frac{1}{\sqrt{\epsilon}}\, K_{\sqrt{1-4\theta\epsilon}}(\sqrt{4\theta})}{\frac{1}{2}\, K_{\sqrt{1-4\theta\epsilon}}(\sqrt{4\theta\epsilon}) - \frac{1}{2}\sqrt{4\theta\epsilon}\, K'_{\sqrt{1-4\theta\epsilon}}(\sqrt{4\theta\epsilon})} \\
&\quad \text{(from (1.5), replacing } \theta \text{ by } \theta\epsilon \text{ and } x \text{ by } \frac{1}{\epsilon}) \\
&= \lim_{\epsilon\downarrow 0} \frac{\frac{1}{\sqrt{\epsilon}}\, K_1(\sqrt{4\theta}) + O(1)}{\frac{1}{2}\, K_{\sqrt{1-4\theta\epsilon}}(\sqrt{4\theta\epsilon})\big[1-\sqrt{1-4\theta\epsilon}\big] + \sqrt{\theta\epsilon}\, K_{\sqrt{1-4\theta\epsilon}+1}(\sqrt{4\theta\epsilon})} \\
&\quad \big(\text{since } \ zK'_\nu(z) = \nu K_\nu(z) - z\, K_{\nu+1}(z); \text{see [Leb], p. 110}\big) \\
&= \lim_{\epsilon\downarrow 0} \frac{\frac{1}{\sqrt{\epsilon}} K_1(\sqrt{4\theta}) + O(1)}{O(1) + \sqrt{\theta\epsilon}\,\big(\frac{1}{2\theta\epsilon} + O(1)\big)} = 2\sqrt{\theta}\, K_1(\sqrt{4\theta})
\end{aligned}
$$

■

2.5 Proof of Theorem 1.4

From Theorem 1.3, we know that, for $\theta \geq \dfrac{1}{4}$:

$$
\begin{aligned}
E_1(e^{-\frac{\theta}{2}\Pi_1}) &= \frac{2K_{i\nu}(\sqrt{4\theta})}{K_{i\nu}(\sqrt{4\theta}) - \sqrt{4\theta}\, K'_{i\nu}(\sqrt{4\theta})} \\
&= \frac{1}{\frac{1}{2} - \sqrt{\theta}\, \frac{K'_{i\nu}(\sqrt{4\theta})}{K_{i\nu}(\sqrt{4\theta})}} = \frac{1}{2}\int_0^\infty e^{-\frac{l}{2}+\frac{l}{2}\left(\frac{1}{2}+\sqrt{\theta}\,\frac{K'_{i\nu}(\sqrt{4\theta})}{K_{i\nu}(\sqrt{4\theta})}\right)} dl
\end{aligned}
\tag{2.20}
$$

On the other hand, from (1.7) :

$$
\begin{aligned}
E_1(e^{-\frac{\theta}{2}\Pi_1}) &= E_1\left(\exp -\frac{\theta}{2}\int_1^\infty \frac{(y-1)^+}{y^2}\,\lambda_{y-1}\, dy\right) \\
&= \frac{1}{2}\int_0^\infty e^{-\frac{l}{2}} dl\, E\left(\exp -\frac{\theta}{2}\int_1^\infty \frac{(y-1)^+}{y^2}\,\lambda_{y-1} dy \big| \lambda_0 = l\right) \\
&= \frac{1}{2}\int_0^\infty e^{-\frac{l}{2}} dl\, Q_l^{(0)}\left(\exp -\frac{\theta}{2}\int_0^\infty \frac{y}{(1+y)^2}\,\lambda_y\, dy\right)
\end{aligned}
\tag{2.21}
$$

where $Q_l^{(0)}$ denotes the expectation relative to a squared Bessel process of dimension 0, starting from l. The comparison of (2.21) and (2.20) implies Theorem 1.4 in the case

$\theta \geq \frac{1}{4}$, since the Laplace transform is one-to-one. The proof of Theorem 1.4 in the case $\theta \leq \frac{1}{4}$ is the same. ■

Remark 2.4 Let μ denote a positive σ-finite measure on $\mathbb{R}_+$. Let Φ denote the unique decreasing, positive solution on $\mathbb{R}_+$ of the Sturm-Liouville equation $\Phi'' = \mu\Phi$ $\big($and such that $\Phi(0) = 1\big)$. It is well known $\big($see [RY], Chap. IX, p. 444$\big)$ that :

$$Q_l^{(0)}\left(\exp\left(-\frac{1}{2}\int_0^\infty \lambda_y\,\mu(dy)\right)\right) = \exp\left(\frac{l}{2}\,\Phi'(0_+)\right) \tag{2.22}$$

$\Big($Observe that, as $(\lambda_y,\ y \geq 0)$ has compact support a.s., $\displaystyle\int_0^\infty \lambda_y\,\mu(dy) < \infty$ if μ is σ-finite on $\mathbb{R}_+\Big)$.

Theorem 1.4 may be recovered easily by applying formula (2.22) with $\mu(dy) = \dfrac{y}{1+y^2}\,dy$. On the other hand, if $\mu(dy) = a(y)dy$, with

$$c_1\,1_{[0,\gamma_1]}(y) \leq a(y) \leq c_2\,1_{[0,\gamma_2]}(y) \qquad (0 < c_1 \leq c_2,\ 0 < \gamma_1 \leq \gamma_2),$$

we deduce from [RY], Chap. XI, Corollary 1.8, that :

$$\exp -\frac{l}{2}\,c_2'\sqrt{\theta} \leq E_l^{(0)}\big(e^{-\frac{\theta}{2}\int_0^\infty \lambda_y\,\mu(dy)}\big) \leq \exp -\frac{l}{2}\,c_1'\sqrt{\theta} \tag{2.23}$$

for θ large enough, whereas, as we shall show in Section 3 :

$$Q_l^{(0)}\left(\exp -\frac{\theta}{2}\int_0^\infty \frac{y}{1+y^2}\,\lambda_y\,dy\right) \underset{\theta\to\infty}{\sim} C\,e^{-\frac{l}{2}\,\theta^{\frac{1}{3}}} \tag{2.24}$$

3 Asymptotic behavior of $E_1(e^{-\frac{\theta}{2}\Pi_1})$ as $\theta \to \infty$

We shall now end the proof of Theorem 1.2 by showing :

Theorem 3.1 *There is the equivalence result:*

$$E_1\big(e^{-\frac{\theta}{2}\Pi_1}\big) \underset{\theta\to\infty}{\sim} \frac{C}{\theta^{\frac{1}{3}}} \tag{3.1}$$

Proof of Theorem 3.1

We recall that, from Theorem 1.3 :

$$E_1\big(e^{-\frac{\theta}{2}\Pi_1}\big) = \frac{2K_{i\nu}(\sqrt{4\theta})}{K_{i\nu}(\sqrt{4\theta}) - \sqrt{4\theta}\,K'_{i\nu}(\sqrt{4\theta})} \tag{3.2}$$

with $\nu = \sqrt{4\theta-1}\big(\theta \geq \frac{1}{4}\big)\cdot$ We shall successively find an equivalent of the numerator and the denominator of (3.2), the difficulty arising from the fact that, in $K_{i\nu}(\sqrt{4\theta})$ $\big($and

$K'_{i\nu}(\sqrt{4\theta})\big)$ the argument $\sqrt{4\theta}$ and index $i\nu = i\sqrt{4\theta - 1}$ tend both to infinity as $\theta \to \infty$. To overcome this difficulty, we shall use some results about Bessel functions found in Watson $\big($[Wat], p. 245-248$\big)$, which we now recall.

i) Let $H^{(1)}_{i\nu}$ the first Hankel function $\big($see [Leb], p. 120$\big)$; it is related to $K_{i\nu}$ via the formula :

$$K_{i\nu}(z) = \frac{i\pi}{2}\, e^{-\frac{\nu\pi}{2}}\, H^{(1)}_{i\nu}(z\, e^{\frac{i\pi}{2}}) \tag{3.3}$$

We define ϵ by the formula : $i\nu = iz(1-\epsilon)$ and assume that, as $z \to \infty$, ϵ remains bounded (with, of course, ν depending on z). Then, there is the second order asymptotic expansion :

$$H^{(1)}_{\nu}(z) = -\frac{2}{3\pi}\left\{ e^{\frac{2}{3}\pi i}\left(\sin\frac{\pi}{3}\right)\frac{\Gamma\left(\frac{1}{3}\right)}{\left(\frac{1}{6}\,z\right)^{\frac{1}{3}}} + e^{\frac{4}{3}\pi i}(\epsilon z)\left(\sin\frac{2\pi}{3}\right)\frac{\Gamma\left(\frac{2}{3}\right)}{\left(\frac{1}{6}\,z\right)^{\frac{2}{3}}} + o\left(\frac{1}{z^{\frac{2}{3}}}\right)\right\} \tag{3.4}$$

ii) Let us study the numerator of (3.2) :

$$\begin{aligned} N &= 2K_{i\nu}(\sqrt{4\theta}) = 2\,\frac{i\pi}{2}\, e^{-\frac{\nu\pi}{2}} H^{(1)}_{i\nu}\big(e^{\frac{i\pi}{2}}\sqrt{4\theta}\big) \\ &= (i\pi)e^{-\frac{\nu\pi}{2}}\left(-\frac{2}{3\pi}\right)\left\{ C_1\, e^{\frac{2}{3}\pi i}\frac{1}{\big(e^{\frac{i\pi}{2}}\sqrt{4\theta}\big)^{\frac{1}{3}}} + o\left(\frac{1}{\theta^{\frac{1}{6}}}\right)\right\} \qquad (3.5)\\ &\qquad\qquad \left(\text{with } C_1 = \left(\sin\frac{\pi}{3}\right)\Gamma\left(\frac{1}{3}\right) 6^{\frac{1}{3}}\ \right) \\ &\underset{\theta\to\infty}{\sim} \frac{2^{\frac{2}{3}}}{3}\, C_1\, e^{-\frac{\nu\pi}{2}}\,\frac{1}{\theta^{\frac{1}{6}}} \qquad (3.6) \end{aligned}$$

Here, we have used the first order expansion (3.4) with : $e^{\frac{2}{3}i\pi}\cdot e^{-\frac{i\pi}{6}} = i \ \left(\frac{4\pi}{6} - \frac{\pi}{6} = \frac{3\pi}{6} = \frac{\pi}{2}\right)$ and the fact that here

$$\begin{aligned} \epsilon z &= z - \nu \\ &= i\sqrt{4\theta} - i\sqrt{4\theta - 1} = i\sqrt{4\theta}\left(1 - \sqrt{1 - \frac{1}{4\theta}}\right) \underset{\theta\to\infty}{\longrightarrow} 0 \end{aligned}$$

iii) We now study the denominator of (3.2)

$$\begin{aligned} D &= K_{i\nu}(\sqrt{4\theta}) - \sqrt{4\theta}\, K'_{i\nu}(\sqrt{4\theta}) \\ &= K_{i\nu}(\sqrt{4\theta}) + \sqrt{4\theta}\, K_{i\nu-1}(\sqrt{4\theta}) + i\sqrt{4\theta - 1}\, K_{i\nu}(\sqrt{4\theta}) \qquad (3.7) \end{aligned}$$

$\big($after using $\frac{d}{dz}\big(z^{\mu}K_{\mu}(z) = -z^{\mu}K_{\mu-1}(z)\big)$; [Leb], p. 110$\big)$. Since we have already studied the asymptotic behavior of $K_{i\nu}(\sqrt{4\theta})$, it remains to study that of

$$\Delta(\theta) := \sqrt{4\theta}\, K_{i\nu-1}(\sqrt{4\theta}) + i\sqrt{4\theta - 1}\, K_{i\nu}(\sqrt{4\theta}) \tag{3.8}$$

Now, since $\quad \sqrt{4\theta}\, K_{i\nu-1}(\sqrt{4\theta}) + i\sqrt{4\theta-1}\, K_{i\nu}(\sqrt{4\theta}) \quad$ is real $\big($from (2.18)$\big)$
and since $i\sqrt{4\theta-1}\, K_{i\nu}(\sqrt{4\theta})$ is purely imaginary,
the development of $\Delta(\theta)$ as $\theta \to \infty$ is that of the real part of $\sqrt{4\theta}\, K_{i\nu-1}(\sqrt{4\theta})$. Then, we obtain :

$$\begin{aligned}
\sqrt{4\theta}\, K_{i\nu-1}(\sqrt{4\theta}) &= \sqrt{4\theta}\, \frac{i\pi}{2}\, e^{\frac{i\pi}{2}(i\nu-1)} H^{(1)}_{i\nu-1}\big(e^{\frac{i\pi}{2}}\sqrt{4\theta}\big) \\
&= \sqrt{4\theta}\, \frac{\pi}{2}\, e^{-\frac{\nu\pi}{2}}\, H^{(1)}_{i\nu-1}\big(e^{\frac{i\pi}{2}}\sqrt{4\theta}\big) \qquad (3.9)
\end{aligned}$$

from (3.3), by replacing $i\nu$ by $i\nu-1$. This time, we use the second order development (3.4) with here : $z = e^{\frac{i\pi}{2}}\sqrt{4\theta}$ and :

$$\begin{aligned}
i\nu - 1 = z(1-\epsilon) &= i\sqrt{4\theta}(1-\epsilon), \text{ i.e. :} \qquad \epsilon z = \epsilon e^{\frac{i\pi}{2}}\sqrt{4\theta} \\
\text{or} \qquad \epsilon z &= \epsilon e^{\frac{i\pi}{2}}\sqrt{4\theta} = 1 + \frac{1}{4\sqrt{\theta}} + o\left(\frac{1}{\sqrt{\theta}}\right) \quad \big(\nu = \sqrt{4\theta-1}\big) \qquad (3.10)
\end{aligned}$$

We obtain :

$$\sqrt{4\theta}\, K_{i\nu-1}(\sqrt{4\theta}) = \sqrt{4\theta}\, \frac{\pi}{2}\, e^{-\frac{\nu\pi}{2}} \left(-\frac{2}{3}\pi\right) \times \cdots$$

$$\cdots \left\{ C_1\, e^{\frac{2i\pi}{3}} \frac{1}{\big(e^{\frac{i\pi}{2}}\sqrt{4\theta}\big)^{\frac{1}{3}}} + C_2\, e^{\frac{4i\pi}{3}} \left(1 + \frac{i}{4\sqrt{\theta}}\right) \cdot \frac{1}{\big(e^{\frac{i\pi}{2}}\sqrt{4\theta}\big)^{\frac{2}{3}}} + o\left(\frac{1}{\theta^{\frac{1}{3}}}\right) \right\} \qquad (3.11)$$

with $C_1 = \left(\sin\frac{\pi}{3}\right)\Gamma\left(\frac{1}{3}\right) 6^{\frac{1}{3}}$ and $C_2 = \left(\sin\frac{2\pi}{3}\right)\Gamma\left(\frac{2}{3}\right) 6^{\frac{2}{3}} \quad (C_1, C_2 > 0)$

$$= -\frac{1}{3}\sqrt{4\theta}\, e^{-\frac{\nu\pi}{2}} \left\{ C_1\, i\, \frac{1}{(\sqrt{4\theta})^{\frac{1}{3}}} - C_2\left(1 + \frac{i}{4\sqrt{\theta}}\right) \frac{1}{(\sqrt{4\theta}^{\frac{2}{3}}} + o\left(\frac{1}{\theta^{\frac{1}{3}}}\right) \right\} \qquad (3.12)$$

Considering now the real part of (3.12) to obtain :

$$\begin{aligned}
\Delta(\theta) &= \mathcal{R}e\big(\sqrt{4\theta}\, K_{i\nu-1}(\sqrt{4\theta}) + i\sqrt{4\theta-1}\, K_{i\nu}(\sqrt{4\theta})\big) \\
&= \frac{1}{3}\sqrt{4\theta}\, e^{-\frac{\nu\pi}{2}} C_2 \left(\frac{1}{(\sqrt{4\theta})^{\frac{2}{3}}} + o\left(\frac{1}{\theta^{\frac{1}{3}}}\right)\right) \\
&\underset{\theta\to\infty}{\sim} \frac{2^{\frac{1}{3}} C_2}{3}\, e^{-\frac{\nu\pi}{2}} \theta^{\frac{1}{6}} \qquad (3.13)
\end{aligned}$$

- We then gather (3.2), (3.6) and (3.13), we obtain :

$$\begin{aligned}
E_1\big(e^{-\frac{\theta}{2}\Pi_1}\big) &\underset{\theta\to\infty}{\sim} \frac{2\,\frac{2^{\frac{2}{3}}}{3} C_1\, e^{-\frac{\nu\pi}{2}} \frac{1}{\theta^{\frac{1}{6}}}}{\frac{2^{\frac{2}{3}}}{3} C_1\, e^{-\frac{\nu\pi}{2}} \frac{1}{\theta^{\frac{1}{6}}} + \frac{2^{\frac{1}{3}}}{3} C_2\, e^{-\frac{\nu\pi}{2}} \theta^{\frac{1}{6}}} \\
&\underset{\theta\to\infty}{\sim} \frac{2^{\frac{4}{3}} C_1}{C_2}\, \frac{1}{\theta^{\frac{1}{3}}} = \frac{2}{3^{\frac{1}{3}}}\, \frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{2}{3})}\, \frac{1}{\theta^{\frac{1}{3}}} \qquad (3.14)
\end{aligned}$$

from the explicit formulae for C_1 and C_2. This is Theorem 3.1.

Remark 3.2 Using Theorem 1.3, we obtain, for $0 < x \leq 1$:

$$E_x\big(e^{-\frac{\theta}{2}\,\Pi_1}\big) = 1 + x\;\frac{\frac{1}{2}\,K_{i\sqrt{4\theta-1}}(\sqrt{4\theta}) + \sqrt{\theta}\;K'_{i\sqrt{4\theta-1}}(\sqrt{4\theta})}{\frac{1}{2}\,K_{i\sqrt{4\theta-1}}(\sqrt{4\theta}) - \sqrt{\theta}\;K'_{i\sqrt{4\theta-1}}(\sqrt{4\theta})}$$

It now follows easily from (3.6) and (3.13) that :

$$E_x\big(e^{-\frac{\theta}{2}\,\Pi_1}\big) \underset{\theta\to\infty}{\longrightarrow} 1 - x \tag{3.15}$$

Now, on the other hand :

$$E_x\big(e^{-\frac{\theta}{2}\,\Pi_1}\big) \underset{\theta\to\infty}{\longrightarrow} P_x(\Pi_1 = 0) \tag{3.16}$$

We recall (and recover here) that $P_x(\Pi_1 = 0) = 1 - x$. This is point *ii)* of Theorem 1.2.

4 Extending the preceding results to the variables $\Pi_k^{(\rho,x)}$ with

$$\Pi_k^{(\rho,x)} := \int_0^\infty \big(e^{\rho(\log\, x + B_t - \frac{t}{2})} - k\big)^+ dt \tag{4.1}$$

$(\rho, x, k > 0)$. In the preceding Sections, we have studied the case $\rho = 1$.
The analogue of Theorem 1.3 may be stated as :
Theorem 4.1 *The Laplace transform of* $\Pi_1^{(\rho)}$ *under* P_x *is given by :*

$$E_x(e^{-\frac{\theta}{2}\,\Pi_1^{(\rho)}})\begin{cases} \dfrac{\sqrt{x}\,K_\gamma\big(\frac{\sqrt{4\theta}}{\rho}\,x^{\frac{\rho}{2}}\big)}{\frac{1}{2}\,K_\gamma(\frac{\sqrt{4\theta}}{\rho}) - \sqrt{\theta}\,K'_\gamma(\frac{\sqrt{4\theta}}{\rho})} & \text{if } x \geq 1 \\[2ex] 1 + x\;\dfrac{\frac{1}{2}\,K_\gamma(\frac{\sqrt{4\theta}}{\rho}) + \sqrt{\theta}K'_\gamma\,(\frac{\sqrt{4\theta}}{\rho})}{\frac{1}{2}\,K_\gamma(\frac{\sqrt{4\theta}}{\rho}) - \sqrt{\theta}\,K'_\gamma(\frac{\sqrt{4\theta}}{\rho})} & \text{if } 0 < x \leq 1 \end{cases} \tag{4.2}$$

with $\gamma = \dfrac{\sqrt{1-4\theta}}{\rho}$ if $\theta \leq \dfrac{1}{4}$ and $\gamma = i\,\dfrac{\sqrt{4\theta-1}}{\rho}$ if $\theta > \dfrac{1}{4}$.
The proof of this Theorem 4.1 is quite similar to that of Theorem 1.3. It hinges upon the following Lemma 4.2.

Lemma 4.2
i) Let, for $\theta \geq 0$ and $\rho > 0$, the function $\varphi_\theta^{(\rho)} : \mathbb{R}_+ \to \mathbb{R}$ defined by :

$$\varphi_\theta^{(\rho)}(y) = \sqrt{y}\, K_\gamma \left(\frac{\sqrt{4\theta}}{\rho}\, y^{\frac{\rho}{2}} \right) \qquad (y \geq 0) \tag{4.3}$$

Then, it satisfies :

$$\varphi_\theta^{(\rho)''}(y) + \theta \left(-y^{\rho-2} + \frac{1}{y^2} \right) \varphi_\theta^{(\rho)}(y) = 0 \tag{4.4}$$

(see [Pet], p. 306, with $\rho - 2 = m$, formula (8), after taking care of replacing a by $-a$).
ii) Let $\widetilde{\varphi}_\theta^{(\rho)} : \mathbb{R}_+ \to \mathbb{R}$ defined by :

$$\widetilde{\varphi}_\theta^{(\rho)}(y) = \begin{cases} \varphi_\theta^{(\rho)}(y) & \text{if } y \geq 1 \\ \varphi_\theta^{(\rho)}(1) - \varphi_\theta^{(\rho)'}(1) + y\, \varphi_\theta^{(\rho)'}(1) & \text{if } y \leq 1 \end{cases} \tag{4.5}$$

Then $\widetilde{\varphi}_\theta^{(\rho)}$ is positive, decreasing, convex and satisfies :

$$\widetilde{\varphi}_\theta^{(\rho)}(y) + \theta \left(-y^{\rho-2} + \frac{1}{y^2} \right) 1_{y \geq 1}\, \widetilde{\varphi}_\theta^{(\rho)}(y) = 0 \tag{4.6}$$

Remark 4.1 As a check, we note that formula (4.2) allows to recover the identity (1.8) :

$$\int_0^\infty e^{\rho B_t - \frac{\rho t}{2}} dt \overset{\text{(law)}}{=} \frac{2}{\rho^2 \gamma_{\frac{1}{\rho}}}.$$

Indeed, on one hand :

$$E\big(e^{-\frac{\theta}{2} \frac{2}{\rho^2 \gamma_{\frac{1}{\rho}}}}\big) = \frac{1}{\Gamma\big(\frac{1}{\rho}\big)} \int_0^\infty e^{-\frac{\theta}{\rho^2 z} - z}\, z^{\frac{1}{\rho}-1} dz = \frac{2}{\Gamma\big(\frac{1}{\rho}\big)}\, K_{\frac{1}{\rho}} \left(\frac{\sqrt{4\theta}}{\rho} \right) \left(\frac{\theta}{\rho^2} \right)^{\frac{1}{2\rho}} \text{(see [Leb], p. 115)}$$

On the other hand :

$$\begin{aligned} E_1\big(e^{-\frac{\theta}{2} \Pi_0^{(\rho)}}\big) &= \lim_{\epsilon \downarrow 0} E_1 \left(\exp -\frac{\theta}{2} \int_0^\infty \big(e^{\rho B_t - \frac{\rho t}{2}} - \varepsilon\big)^+ dt \right) \\ &= \lim_{\epsilon \downarrow 0} E_1 \left(\exp -\frac{\theta \varepsilon}{2} \int_0^\infty \big(e^{\rho\left(\frac{\log \frac{1}{\varepsilon}}{\rho} + B_t\right) - \frac{\rho t}{2}} - 1\big)^+ dt \right) \\ &= \lim_{\epsilon \downarrow 0} \frac{\big(\frac{1}{\varepsilon}\big)^{\frac{1}{2\rho}} K_{\frac{\sqrt{1-4\theta\varepsilon}}{\rho}} \big(\frac{\sqrt{4\theta\varepsilon}}{\rho}\big) \big(\frac{1}{\varepsilon}\big)^{\frac{1}{\rho}\frac{\rho}{2}}}{\frac{1}{2} K_{\frac{\sqrt{1-4\theta\varepsilon}}{\rho}} \big(\frac{\sqrt{4\theta\varepsilon}}{\rho}\big) - \sqrt{\theta\varepsilon}\, K'_{\frac{\sqrt{1-4\theta\varepsilon}}{\rho}} \big(\frac{\sqrt{4\theta\varepsilon}}{\rho}\big)} \end{aligned}$$

from (4.2), replacing θ by $\theta\varepsilon$ and x by $\left(\dfrac{1}{\varepsilon}\right)^{\frac{1}{\rho}}$

$$
\begin{aligned}
&\underset{\varepsilon\to 0}{\sim} \quad \frac{\left(\frac{1}{\varepsilon}\right)^{\frac{1}{2\rho}} K_{\frac{1}{\rho}}\left(\frac{\sqrt{4\theta}}{\rho}\right)}{\frac{1}{2}\, K_{\frac{\sqrt{1-4\theta\varepsilon}}{\rho}}\left(\frac{\sqrt{4\theta\varepsilon}}{\rho}\right) - \frac{\rho}{2}\left(\frac{\sqrt{4\theta\varepsilon}}{\rho}\right) K'_{\frac{\sqrt{1-4\theta\varepsilon}}{\rho}}\left(\frac{\sqrt{4\theta\varepsilon}}{\rho}\right)} \\
&\underset{\varepsilon\to 0}{\sim} \quad \frac{\left(\frac{1}{\varepsilon}\right)^{\frac{1}{2\rho}} K_{\frac{1}{\rho}}\left(\frac{\sqrt{4\theta}}{\rho}\right)}{\frac{\rho}{2}\Big[\frac{\sqrt{4\theta\varepsilon}}{\rho}\, K_{\frac{1}{\rho}+1}\left(\frac{\sqrt{4\theta\varepsilon}}{\rho}\right)} \qquad \Big(\text{since } zK'\nu = \nu K_\nu(z) - z\, K_{\nu+1}(z)\Big) \\
&\underset{\varepsilon\to 0}{\sim} \quad \frac{\left(\frac{1}{\varepsilon}\right)^{\frac{1}{2\rho}} K_{\frac{1}{\rho}}\left(\frac{\sqrt{4\theta}}{\rho}\right)}{\frac{\rho}{2}\Big(\Gamma(\frac{1}{\rho}+1)\left(\frac{\sqrt{4\theta\varepsilon}}{\rho}\right)^{-\frac{1}{\rho}}} \qquad \left(\text{since } K_\nu(z) \sim \frac{1}{2}\,\Gamma(\nu)\left(\frac{z}{2}\right)^{-1}\right) \\
&= \quad \frac{\left(\frac{1}{\varepsilon}\right)^{\frac{1}{2\rho}} K_{\frac{1}{\rho}}\left(\frac{\sqrt{4\theta}}{\rho}\right)}{\Gamma\left(\frac{1}{\rho}\right)} \; \frac{\varepsilon^{\frac{1}{2\rho}}\theta^{\frac{1}{2\rho}}}{\rho^{\frac{1}{\rho}}} \\
&\underset{\varepsilon\downarrow 0}{\longrightarrow} \quad \left(\frac{\theta}{\rho^2}\right)^{\frac{1}{2\rho}} \frac{2}{\Gamma\left(\frac{1}{\rho}\right)}\, K_{\frac{1}{\rho}}\left(\frac{\sqrt{4\theta}}{\rho}\right) = E\left(e^{-\frac{\theta}{2}\Pi_0^{(\rho)}}\right)
\end{aligned}
$$

Remark 4.2
Taking up again the arguments of the proof of Theorem 3.1, it is not difficult to see that :

$$E_1\left(e^{-\frac{\theta}{2}\Pi_1^{(\rho)}}\right) \underset{\theta\to\infty}{\sim} \frac{C(\rho)}{\theta^{\frac{1}{3}}}$$

where $C(\rho)$ is a strictly positive constant, depending on ρ. We then deduce that, for $\alpha < 0$:

$$E_1\left((\Pi_1^{(\rho)})^\alpha\right) < \infty \text{ if and only if } |\alpha| < \frac{1}{3}$$

On the other hand, it is not difficult to see that :

- If $x > 1$, for all $\alpha < 0$, $E_1\left((\Pi_1^{(\rho)})^\alpha\right) < \infty$.
- If $x < 1$, for all $\alpha < 0$, $E_1\left((\Pi_1^{(\rho)})^\alpha\right) = +\infty$.

Concerning the positive moments of $\Pi_1^{(\rho)}$:

$$\bullet \qquad If \quad 0 < \alpha < \frac{1}{\rho}, \quad \text{then} \quad E_x\left((\Pi_1^{(\rho)})^\alpha\right) < \infty \tag{4.7}$$

Indeed (4.7), for $x = 1$, follows from :

$$
\begin{aligned}
E_1\left((\Pi_1^{(\rho)})^\alpha\right) \;&\leq\; E_1\left((\Pi_0^{(\rho)})^\alpha\right) = \frac{1}{\Gamma\left(\frac{1}{\rho}\right)} \int_0^\infty \left(\frac{2}{\rho^2 z}\right)^\alpha e^{-z} z^{\frac{1}{\rho}-1} dz \\
&<\; \infty \quad \text{if} \quad \alpha < \frac{1}{\rho}
\end{aligned}
$$

$$\left(\text{since} \qquad \Pi_0 \overset{\text{(law)}}{=} \frac{2}{\rho^2 \gamma_{\frac{1}{\rho}}}, \quad \text{from (1.10)}\right)$$

The fact that $E_x\big((\Pi_1^{(\rho)})^\alpha\big) < \infty$ for every $x > 0$, and every $\alpha < \dfrac{1}{\rho}$ may be obtained by using arguments close to those used in the proof of Theorem 1.1.
We believe that, for all $\rho > 0$:

$$E_x\big((\Pi_1^{(\rho)})^{\frac{1}{\rho}}\big) = +\infty. \tag{4.8}$$

We now show (4.8), when $\rho > 1$

It then suffices, by using the arguments of the proof of Theorem 1.1, to see that (4.8) is true when $x = 1$.

- We first show that : $1 - E_1\big(e^{-\frac{\theta}{2}\Pi_1^{(\rho)}}\big) \underset{\theta\to 0}{\sim} C\theta^{\frac{1}{\rho}}$ (4.9)

Indeed, from Theorem 4.1, we have :

$$E_1\left(\exp -\frac{\theta}{2}\,\Pi_1^{(\rho)}\right) = 1 + \frac{\frac{1}{2}\,K_\gamma\big(\frac{\sqrt{4\theta}}{\rho}\big) + \sqrt{\theta}\,K'_\gamma\big(\frac{\sqrt{4\theta}}{\rho}\big)}{\frac{1}{2}\,K_\gamma\big(\frac{\sqrt{4\theta}}{\rho}\big) - \sqrt{\theta}\,K'_\gamma\big(\frac{\sqrt{4\theta}}{\rho}\big)}$$

with $\gamma = \dfrac{\sqrt{1-4\theta}}{\rho}$ $\big($and $\theta \le \dfrac{1}{4}\big)\cdot$ Thus :

$$\begin{aligned}
1 - E_1\big(e^{-\frac{\theta}{2}\Pi_1^{(\rho)}}\big) &= -\frac{\frac{1}{2}\,K_\gamma\big(\frac{\sqrt{4\theta}}{\rho}\big) + \frac{\rho}{2}\big\{\frac{\sqrt{4\theta}}{\rho}\,K'_\gamma\big(\frac{\sqrt{4\theta}}{\rho}\big)\big\}}{\frac{1}{2}\,K_\gamma\big(\frac{\sqrt{4\theta}}{\rho}\big) - \frac{\rho}{2}\big\{\frac{\sqrt{4\theta}}{\rho}\,K'_\gamma\big(\frac{\sqrt{4\theta}}{\rho}\big)\big\}} \\
&= -\frac{\frac{1}{2}\,K_{\frac{\sqrt{1-4\theta}}{\rho}}\big(\frac{\sqrt{4\theta}}{\rho}\big)\big(1 - \frac{\sqrt{1-4\theta}}{\rho}\big) - \frac{\sqrt{\theta}}{\rho}\,K_{\frac{\sqrt{1-4\theta}}{\rho}-1}\big(\frac{\sqrt{4\theta}}{\rho}\big)}{\frac{1}{2}\,K_{\frac{\sqrt{1-4\theta}}{\rho}}\big(\frac{\sqrt{4\theta}}{\rho}\big)\big(1 + \frac{\sqrt{1-4\theta}}{\rho}\big) + \frac{\sqrt{\theta}}{\rho}\,K_{\frac{\sqrt{1-4\theta}}{\rho}-1}\big(\frac{\sqrt{4\theta}}{\rho}\big)}
\end{aligned}$$

$\big($after using $zK'_\mu(z) = -\mu\,K_\mu(z) - z\,K_{\mu-1}(z)\big)$.
Since $\rho > 1$, we replace $K_{\frac{\sqrt{1-4\theta}}{\rho}-1}$ by $K_{1-\frac{\sqrt{1-4\theta}}{\rho}}$ (since $K_\mu = K_{-\mu}$) and we deduce, from :

$$K_{1-\frac{\sqrt{1-4\theta}}{\rho}} \underset{\theta\to 0}{\longrightarrow} K_{1-\frac{1}{\rho}}, \quad \text{with} \quad 1 - \frac{1}{\rho} > 0 \ :$$

that :

$$\begin{aligned}
1 - E_1\big(e^{-\frac{\theta}{2}\Pi_1^{(\rho)}}\big) &\underset{\theta\to 0}{\sim} \frac{a\,\theta^{1-\frac{1}{2\rho}} + b\,\theta^{\frac{1}{2\rho}}}{a'\,\theta^{-\frac{1}{2\rho}} + b'\,\theta^{\frac{1}{2\rho}}} \ \big(\text{since, for } \mu > 0\ K_\mu(z) \underset{z\to 0}{\sim} C_\mu\,z^{-\mu}\ ;\ \text{see [Leb], p. 136}\big) \\
&\underset{\theta\to 0}{\sim} \frac{b}{a'}\,\theta^{\frac{1}{\rho}} \qquad \left(\text{since } \rho \ge 1 \text{ implies } 1 - \frac{1}{2\rho} \ge \frac{1}{2\rho}\right)
\end{aligned}$$

- From (4.9) we deduce :

$$\begin{aligned}
\int_0^\infty e^{-\frac{\theta t}{2}} P(\Pi_1^{(\rho)} \ge t)dt &= \frac{2}{\theta}\big(1 - E(e^{-\frac{\theta}{2}\Pi_1^{(\rho)}})\big) \\
&\underset{\theta\to\infty}{\sim} C'\theta^{\frac{1}{\rho}-1}
\end{aligned}$$

Hence, from the Tauberian Theorem :

$$P(\Pi_1^{(\rho)} \geq t) \underset{t\to\infty}{\sim} C'' \frac{1}{t^{\frac{1}{\rho}}} \tag{4.10}$$

and

$$\begin{aligned} E\big((\Pi_1^{(\rho)})^{\frac{1}{\rho}}\big) &= \int_0^\infty P\big((\Pi_1^{(\rho)})^{\frac{1}{\rho}} \geq t\big) dt \\ &= \int_0^\infty P\big(\Pi_1^{(\rho)} \geq t^\rho\big) dt = +\infty, \qquad \text{from (4.10).} \end{aligned}$$

We note that, for $\rho < 1$, the preceding argument cannot be applied since :

$$\int_1^\infty e^{-\frac{\theta t}{2}} \frac{dt}{t^{\frac{1}{\rho}}} \xrightarrow[\theta\to 0]{} \int_1^\infty \frac{dt}{t^{\frac{1}{\rho}}} < \infty.$$

References

[Duf] D. Dufresne *The distribution of a perpetuity, with applications to risk theory and pension funding.* Scand. Actuarial J., p. 39-79, (1990).

[Kam] E. Kamke *Differentialgleichungen. Lösungsmethoden and Lösungen.* Becker-Erler Verlag, Leipzig, (1942).

[Leb] N.N. Lebedev *Special functions and their applications.* Dover Publ. (1972).

[MRY1] D. Madan, B. Roynette, M. Yor *An alternative expression for the Black-Scholes formula in terms of Brownian first and last passage times.* Prépublication IECN, n° 08, (2008).

[MRY2] D. Madan, B. Roynette, M. Yor *Unifying Black-Scholes type formulae which involve Brownian last passage times up to a finite horizon.* Prépublication IECN, n° 23, (2008). To appear in Asia Pacific Markets Springer.

[MRY3] D. Madan, B. Roynette, M. Yor *Put option prices as joint distribution functions in strike and maturity : the Black-Scholes case.* Prépublication IECN, n° 41, (2008).

[NY] A. Nikeghbali, M. Yor *Doob's maximal identity, multiplicative decomposition and enlargment of filtration.* Illinois J. Maths Special Vol. for J. Doob, p. 791-814, (2005).

[Pet] G. Petiau *La théorie des fonctions de Bessel.* CNRS, (1955).

[RY] D. Revuz, M. Yor *Continuous Martingales and Brownian motion.* Third edition, Springer, Corrected 3rd printing (2005).

[SY1] P. Salminen, M. Yor *Properties of perpetual integral functionals of Brownian motion with drift.* Ann. IHP, 41 (3), p. 335-347, (2005).

[SY2] P. Salminen, M. Yor *Perpetual integral functionals as hitting and occupation times.* Elect. J. Prob., 10, p. 371-419, (2005).

[SY3] P. Salminen, M. Yor *On Dufresne's perpetuities, translated and reflected.* Stoch. proc. and applications to Math. finance, in Proceedings of the 3rd International Symposium Ritsumeikan Univ. World Scientific Publishing Co, Inc., River Edge, p. 337-354, (2004).

[Wat] G.N. Watson *A treatise on the theory of Bessel functions.* Cambridge University Press, Cambridge, England. The Macmillan Company, New-York, (1944).

[Y] M. Yor *On certain exponential functionals of real-valued Brownian motion* . Paper 1, in Exp. functionals of Brownian motion and Related processes, Springer-Finance (2001).